MA5360 – Assignment 1
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1. Prove the reverse triangle inequality: $|w \pm z| \geq ||w| - |z||$ for all $z, w \in \mathbb{C}$ with equality iff either z or w is real multiple of the other.
2. For $z = x + iy$, show that
$$|z| \leq |x| + |y| \leq \sqrt{2}|z|.$$
Also show that the constant $\sqrt{2}$ cannot, in general, be replaced by a smaller constant.
3. Let $T : \mathbb{C} \to \mathbb{C}$ be of the form $\lambda z + \mu \bar{z}$. Show that
   a) T is bijective iff $\lambda\bar{\lambda} \neq \mu\bar{\mu}$.
   b) $|T(z)| = |z|\ \forall z \in \mathbb{C}$ iff $|\lambda + \mu| = 1$ and $\lambda\mu = 0$.
4. For $n > 1$, let $c_0 > c_1 > \cdots > c_n > 0$ be real numbers. Let
$$p(z) := c_0 + c_1 z + \cdots + c_n z^n.$$
Show that every root of p must have modulus greater than 1.
5. At what points of $\mathbb{C}$ is the function $f(z) = z(z + \bar{z}^2)$ $\mathbb{C}$-differentiable?
6. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) := \sqrt{|x||y|}$. Show that f satisfies the Cauchy–Riemann equations at 0. Is f $\mathbb{C}$-differentiable at 0.
7. Let U be a domain and let $f = u + iv$ be holomorphic in U. Suppose for some $v_1 : U \to \mathbb{R}$, we have that $u + iv_1$ is also holomorphic in U. What is the relationship between v and v_1?
8. Let $f : U \to \mathbb{C}$ be real-differentiable in U and suppose
$$\lim_{h \to 0} \left| \frac{f(a+h) - f(a)}{h} \right|$$
exists for the point $a \in U$. Prove that either f or $\bar{f}$ is $\mathbb{C}$-differentiable at a.
9. Show the following identity for the Wirtinger derivatives ∂ and $\bar{\partial}$: $\overline{\partial f} = \bar{\partial}\bar{f}$, $\bar{\partial}\,\bar{f} = \overline{\partial f}$.
10. Write down an expression for the Laplacian of a real-differentiable function $f : U \to \mathbb{C}$ in terms ∂ and $\bar{\partial}$. Do the same for the Jacobian of f. If f is holomorphic in U, what can you tell about the Jacobian of f?
11. Formulate and prove a version of the chain rule for the Wirtinger derivatives.
12. Let $f : U \to \mathbb{C}$ be holomorphic. Show that f is constant if one of $|f|$, $\mathrm{Re} f$ or $\mathrm{Im} f$ is constant.